Solution Set 7

November 2, 2002

1 Problem 1

a).

$$dim(h/e^2) = J * s/C^2$$

From $V = IR$, $I = \frac{dQ}{dt}$ and $V = U/Q$, where U is potential energy we get

$$\Omega = Volt/Amp = (J/C)/(C/s) = J * s/C^2$$

b). $h/e^2 = 2.6 * 10^4 \Omega$
c).

$$\frac{h/e^2}{Z_0} = \frac{h\sqrt{\epsilon_0}}{e^2\sqrt{\mu_0}}$$

2 Problem 2

a). The $n'th$ radius R_n is given by Eqns. $5-9$ $5-10$ and is

$$R_n = \frac{n^2\hbar}{Zmc\alpha},$$

where Z is the charge of the nucleus and m is the mass of the particle in question. In our case and in terms of a_0 the equation can be written as

$$R_1 = \frac{m_e a_o}{Zm_\mu} = \frac{2 * 0.53 * 10^{-10}}{207A} = 1.2 * 10^{-15} A^{1/3}.$$

Thus $A = 94$.

b). A). We have a moving (and noninteracting) particle. The time dilation effects make time pass more quickly in the lab frame than in muon's frame (from the lab's point of view), and therefore it's apparent lifetime will be *bigger*. B). The particle will see the nucleons as soon as it enters the nucleus, and therefore should interact very quickly. Thus its apparent lifetime will be *smaller*.

3 Problem 3

We want to solve equations

$$mv^2/r = |F_0|$$

and

$$mvr = n\hbar$$

in order to find the energy $E = F_0 r + \frac{1}{2}mv^2$ (note that in order for motion to be circular it must hold that $F_0 = -|F_0|$). Thus

$$E_n = |F_0| r_n(-1 + 1/2) = -\frac{1}{2}|F_0|(\frac{n^2\hbar^2}{m|F_0|})^{1/3}$$

4 Problem 4

a). The acceleration of an electron in a *circular* orbit is just the centripetal acceleration, i.e.

$$a = v^2/r = \frac{\alpha^3 c^3 m}{n^4 \hbar}$$

b). Using the fact that $\alpha = e^2/(4\pi\epsilon_0\hbar c$ we get

$$P = \frac{2\alpha^7 c^4 m^2}{3n^8\hbar}$$

c). At the $n'th$ level, the energy is given by Eq. 5-14 to be

$$E = \frac{-\hbar^2}{2ma_0^2 n^2} = -\frac{mc^2\alpha^2}{2n^2}.$$

Thus

$$E_{n+1->n} = \frac{mc^2\alpha^2}{2}(\frac{1}{n^2} - \frac{1}{(n+1)^2}).$$

For large n we can taylor expand $\frac{1}{(n+1)^2}$ in powers of $1/n$ and neglect all terms but the first two (we have to keep two because the leading term will cancel with $1/n^2$). Thus

$$\frac{1}{(n+1)^2} = \frac{1}{n^2}(\frac{1}{(1+1/n)^2}) = \frac{1}{n^2}(1 - 2/n + ...)$$

and we can use this to get

$$E_{n+1->n} = \frac{mc^2\alpha^2}{n^3}$$

d).

$$t_{n+1->n} = E_{n+1->n}/P_n = \frac{3\hbar}{2\alpha^5 c^2 m} n^5$$

e). Plugging in the appropriate values we get $t_{2->1} = 2.8 * 10^{-10} s$.

5 Problem 5

a). For circular orbits the momentum is constant and parallel to $\vec{dr}$, thus

$$\oint \vec{p} \cdot \vec{dr} = mv * 2\pi r = 2\pi n\hbar$$

and the conditions are identical.

b). Now only the energy is constant and $p = \sqrt{2mE - m^2\omega^2 x^2}$. Hence

$$\oint \vec{p} \cdot \vec{dr} = \int \sqrt{2mE - m^2\omega^2 x^2} dx = \sqrt{2mE} \int \sqrt{1 - A^2 x^2} dx,$$

where $A^2 = m\omega^2/(2E)$ and we can evaluate the integral by trig substitution $Ax = \sin\theta$. Then

$$\oint \vec{p} \cdot \vec{dr} = \frac{1}{A\sqrt{2mE}} \int \cos^2\theta d\theta = \frac{\sqrt{2mE}}{A} \int \frac{1}{2}(1 + \cos(2\theta)) d\theta.$$

Since we are integrating over a complete period the limits of θ are 0 and 2π. Thus

$$2\pi n\hbar = \oint \vec{p} \cdot \vec{dr} = \frac{\pi\sqrt{2mE}}{A} = \frac{2\pi E}{\omega}$$

and

$$E = n\hbar\omega$$

c). For this particle there is no potential energy and as a result $|p| = \sqrt{2mE}$. Since the particle travels a distance of $2L$ during a complete cycle

$$\oint \vec{p} \cdot \vec{dr} = 2L\sqrt{2mE} = 2\pi n\hbar$$

and

$$E = \frac{\pi^2\hbar^2 n^2}{2mL^2}$$

6 Problem 6

Here we use the equations on page 577 to compute the integrals. First,

$$< x^2 > = \int \psi^* x^2 \psi dx = C^2 \int_{-\infty}^{\infty} x^2 e^{-x^2/a^2} dx = C^2 \sqrt{\pi} a^3/2.$$

Also, $p = -i\hbar\frac{d}{dx}$ and therefore

$$< p^2 > = -\hbar^2 \int \psi^* \frac{d^2\psi}{dx^2} dx = -\hbar^2 C^2 \int_{-\infty}^{\infty} (-1/a^2 - x/a^2) e^{-x^2/a^2} dx = C^2\hbar^2\sqrt{\pi}/a,$$

where we used the fact that xe^{-x^2/a^2} is an odd function and therefore $\int_{-\infty}^{\infty} xe^{-x^2/a^2}dx = 0$. Furthermore, C can be obtained from the normalization condition; i.e.

$$1 = \int \psi^* \psi dx = C^2 \int_{-\infty}^{\infty} e^{-x^2/a^2} dx = C^2 \sqrt{\pi} a.$$

Combining the equations we get

$$< x^2 >< p^2 >= C^4 \hbar^2 \pi a^2/2 = \hbar^2/2$$

7 Problem 7

a). Once again $p = -i\hbar \frac{d}{dx}$, and therefore

$$I = -e < v >= -e/m < p >= -ie\hbar/m \int_{-\infty}^{\infty} \psi^* \frac{d\psi}{dx} dx.$$

To show that I is real, we must show that $I^* = I$.

$$I^* = ie\hbar/m \int_{-\infty}^{\infty} \psi \frac{d\psi^*}{dx} dx.$$

We now use the fact that $\psi(\infty, t) = \psi(-\infty, t) = 0$, which follows from normalizability, and get

$$0 = \psi(\infty, t)\psi^*(\infty, t) - \psi(-\infty, t)\psi^*(-\infty, t) = \int_{-\infty}^{\infty} \frac{d(\psi^*\psi)}{dx} dx = \int_{-\infty}^{\infty} \psi \frac{d\psi^*}{dx} dx + \int_{-\infty}^{\infty} \psi^* \frac{d\psi}{dx} dx = 0$$

Thus

$$I^* = -ie\hbar/m \int_{-\infty}^{\infty} \psi^* \frac{d\psi}{dx} dx = I$$

b). Using the fact that $1/i = -i$ we get

$$-e < v > /2 - e < v >^* /2 = (I + I^*)/2 = \frac{-e\hbar}{2im} \int_{-\infty}^{\infty} (\psi \frac{d\psi^*}{dx} - \psi^* \frac{d\psi}{dx}) dx.$$

8 Problem 8

Since ψ satisfies the time independent SE,

$$\frac{d^2\psi}{dx^2} = -\frac{2m}{\hbar^2}(E - U(x))\psi$$

and since $U(x)$ is real

$$\frac{d^2\psi^*}{dx^2} = -\frac{2m}{\hbar^2}(E - U(x))\psi^*.$$

We can use these relations to state that

$$\frac{dj}{dx} = \frac{d^2\psi^*}{dx^2}\psi - \frac{d^2\psi}{dx^2}\psi^*$$

can also be written as

$$\frac{dj}{dx} = -\frac{2m}{\hbar^2}(E - U(x))\psi^*\psi + \frac{2m}{\hbar^2}(E - U(x))\psi\psi^* = 0.$$